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Spinor structures on flag manifolds of compact simple Lie groups

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Abstract. In this paper spinor structures over flag manifolds of compact simple Lie groups are considered and constructed explicitly, using the general method of Dąbrowski and Trautman. In this way the existence and uniqueness of these structures is established, in accordance with purely topological results of Freed. Application of the structures for further studies of fermionic excitations in theories with coadjoint orbits as phase spaces, including infinite-dimensional systems such as superfluid helium, is suggested.

1. Introduction

Many systems of interest in physics have their configuration spaces in the form of Lie groups. For example, a free rigid body has as its configuration space the $SO(3)$ Lie group. The most interesting example for us is the incompressible non-viscous fluid, for which the configuration space is *SDiff* M, the group of diffeomorphisms of a smooth manifold M. The manifold is a region in the space $\mathbb{R}^3$ (or $\mathbb{R}^2$ for two-dimensional flows). This group of diffeomorphisms is infinite-dimensional and as such possesses weaker properties than its finite-dimensional counterparts, in particular it is so-called Frechét–Lie group (for more details see [1, 2]). There are also many other interesting systems with Lie groups as configuration spaces considered in the literature, e.g. [3–6]. The high symmetry causes existence of many integrals of motion and this sometimes leads to complete integrability of these systems.

Two facts motivated our research. The first was the existence of fermionic degrees of freedom connected with the motion of critical superfluid helium [7]. Superfluid helium is a substance that could be modelled as a quantized non-viscous incompressible fluid. Its classical configuration space Q is identified with *SDiff* M, defined above. Due to right invariance of the Euler equations with respect to the relabelling of particles by an element of *SDiff* M, the classical phase space for the motion of the fluid is reduced from the cotangent bundle T^*Q to a coadjoint orbit of the group *SDiff* M. In our previous papers [7–9] we have shown that there are some spinor fields associated with coadjoint orbits of particularly interesting vortex structures. Our approach was based on using Feynman path integrals. This approach, although successful in explaining the heat capacity dependence on temperature for critical superfluid helium [9], is rather heuristic and is not very well founded mathematically. It is reasonable to also look for the heuristically introduced spinor fields in the geometric

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description of the fluid. Namely, it is well grounded to search for spinor bundles and spinor fields connected with the coadjoint orbits. The second important fact that caused our interest in the spinor structures is the theorem which can be found, for example, in the paper by Freed [10] that spinor structures exist for all generic, i.e. of maximal dimension, coadjoint orbits of compact semisimple Lie groups (finite dimensional). Such orbits are called flag manifolds.

Moreover, it could be interesting to find links between the approach and existing results on quantization of superfluid helium within the coadjoint orbits framework represented by Goldin *et al* [11, 12] and also by Penna and Spera [13, 14] and within canonical quantization by Rasetti and Regge [15].

The general programme we can propose consists of the following tasks.

(a) Establishing conditions under which there exist spinor structures on the coadjoint orbits for the group *SDiff* M.
(b) Explicit construction of the spinor structures and spinor fields.
(c) Physical interpretation of the fermions, their connection with vortices and, especially, with topologically non-trivial vortices.

This programme is rather difficult to be realized completely. Before treating the full programme we investigate in this paper spinor structures on coadjoint orbits for finite-dimensional simple compact Lie groups. Although this problem is simpler than the infinite-dimensional one, the fact that $SU(N)$ groups approximate the group *SDiff* (T^2) in the limit of N going to infinity [16, 17] gives us hope that at least some of the constructed spinor structures will help us to understand the general case. We also proved recently that the Euler equations for the $SU(N)$ approximate these for *SDiff* T^2 [18], a fact first anticipated by Zeitlin [19].

The theorem on existence of spinor structures mentioned above, being purely topological, does not give explicit construction of the spinor structures. Such an opportunity gives the theorem proved by Dąbrowski and Trautman [20] which shows a relatively simple group-theoretic way of considering spinor structures over homogeneous Riemannian manifolds. Flag manifolds with fixed complex structures (such structures exist for coadjoint orbits of finite-dimensional groups) are homogeneous Riemannian and the theorem could be applied in this case.

The paper is organized as follows. Firstly, the basic facts about spinor structures on homogeneous Riemannian manifolds are shortly reviewed. Secondly, the geometry of generic coadjoint orbits of simple compact Lie groups needed for further consideration is investigated. Thirdly, spinor structures on the flag manifolds are constructed and some simple examples are presented in detail. Finally, conclusions and prospects for further investigations are shown.

2. Spinor structures on homogeneous Riemannian manifolds

The Riemannian manifold is a manifold M equipped with a metric tensor g, i.e. with a symmetric non-degenerate covariant tensor of second rank with Euclidean signature. Existence of the metric tensor is equivalent to reduction of the natural frame bundle from $Gl(n, \mathbb{R})$ to $O(n)$ ($n = \dim M$), possible for all paracompact manifolds. For orientable manifolds further reduction to the $SO(n)$ group is possible. On the other hand, spinor structures are not natural bundles and there exist topological obstacles for their existence. Spinor structures are twofold covers of the orthonormal $SO(n)$ frame bundles. Their existence is guaranteed by vanishing of the first and second Stiefel–Whitney classes w_1 and w_2. Calculation of these Stiefel–Whitney classes is usually difficult. Moreover, it does not lead to an explicit form of the bundles.

Therefore, it is reasonable to look for more geometric and algebraic instead of topological methods to derive the results, at least for some class of manifolds.

There is an approach of this kind which enables examination of the problem of existence of spinor structures on manifolds that are homogeneous Riemannian, introduced by Dąbrowski and Trautman [21, 20]. Let us restrict all further discussion to smooth and compact manifolds. A homogeneous Riemannian manifold is a Riemannian manifold on which a Lie group G acts transitively by isometries. As a result the manifold M is diffeomorphic to the quotient space G/H where H is a subgroup of G which is a subgroup of isotropy of a point in M. The form of the orthonormal frame bundle over the manifold $M \simeq G/H$ was derived in [21]. It reads:

$$\begin{array}{l} G/N \leftarrow H/N \\ \downarrow \\ M \simeq G/H \end{array} \tag{1}$$

where N is a subgroup of ineffectiveness of the action of G in the tangent space of a point in M. Since H/N is obviously a subgroup of $O(n)$ ($n = \dim M$) and in the orientable case of $SO(n)$, the spinor structure on M would be a bundle double covering G/N, with its structure group being a double cover of H/N that is a counterimage of H/N for the usual covering map $\rho_n\colon \mathrm{Spin}(n) \to SO(n)$. The scheme of constructing the bundle is as follows [20]. We begin with constructing the universal covering space for G/N with the structural group $\Pi_1(G/N)$:

$$\begin{array}{l} \widetilde{G/N} \leftarrow \Pi_1(G/N) \\ \downarrow \\ G/N. \end{array} \tag{2}$$

One should consider the homomorphisms: $h\colon \Pi_1(G/N) \to \mathbb{Z}_2$. Then one should construct bundles associated with the universal covering bundle for G/N by homomorphisms h:

$$\begin{array}{l} (G/N)_h \leftarrow \mathbb{Z}_2 \\ \sigma \ \downarrow \\ G/N. \end{array} \tag{3}$$

After that one should compare the groups $\sigma^{-1}(H/N)$ and $\rho_n^{-1}(H/N)$, where σ is the covering map. If they are isomorphic, the bundle

$$\begin{array}{l} (G/N)_h \leftarrow \sigma^{-1}(H/N) \\ \downarrow \\ M \simeq G/H \end{array} \tag{4}$$

is the spinor structure we were looking for.

Actually in the general case the spinor structure could not exist or there could be more than one such structure. Since the generic coadjoint orbits, which we consider, are simply connected these structures should be unique, and by a theorem [10] they always exist in these particular cases. Therefore, if we find one possible candidate for a spinor structure with our method, it is certainly it, and we should not check whether $\sigma^{-1}(H/N)$ is isomorphic to $\rho_n^{-1}(H/N)$. On the other hand, we should check the condition whenever our method provides us with more than one such candidate. Then only the one which satisfies the condition leads to the spinor structure. Practically, here we do not check the isomorphism of the groups directly. These groups are, as a rule, of quite a simple form, and there are some specific topological characteristics, which distinguish them, e.g. whenever the total space of the candidate spinor structure consists of two copies of the same simply connected space, so that it is a direct product of this space by $\mathbb{Z}_2$, then the structure group should also be of analogous form. On the other hand, if the candidate total space is simply connected, so should the structure group be. The reason in either of these cases is that the base manifold is simply connected. Of course, more subtle situations could, occur in principle, but we have found in all discussed cases that such argumentation is sufficient to eliminate all the candidates for spinor structures, which are not the one we were looking for, and we were left with only one case, which had to be the one searched for.

3. Geometry of coadjoint orbits of simple compact Lie groups

As we mentioned in the introduction, we are interested mainly in spinor structures on coadjoint orbits of Lie groups. We will investigate this problem for the case of simple compact Lie groups.

Coadjoint orbits result from coadjoint action of a Lie group in the dual of its Lie algebra. Let $Ad_g\colon \mathfrak{G} \to \mathfrak{G}$ be the usual adjoint action of the group G in its Lie algebra $\mathfrak{G}$ defined for the element g$\in G$. Let $\langle\cdot\,,\cdot\rangle\colon \mathfrak{G}^* \times \mathfrak{G} \to \mathbb{R}$ be the non-degenerate pairing. The coadjoint action is defined by

$$\rho(g)\colon \quad \mathfrak{G}^* \to \mathfrak{G}^* \qquad g \in G \qquad \rho(g) = Ad^*_{g-1}$$

where $\langle Ad^*_{g^{-1}}\mu, x\rangle \equiv \langle \mu, Ad_{g^{-1}}x\rangle$ for all $\mu \in \mathfrak{G}^*$, $x \in \mathfrak{G}$.

For this action the isotropy subgroup H is isomorphic in the finite-dimensional compact generic case to the maximal torus T of the group G, $H \simeq T$, and the orbits can be identified with the quotient spaces G/T. These orbits are naturally equipped with the Kirillov–Kostant–Souriau (KKS) symplectic structure (the same is also true for the orbits of infinite-dimensional groups). For every almost complex structure that agrees with the KKS form existing on the orbit, the number of which is equal to the number of elements of the Weyl group of G, there is a Riemannian metric on the orbit. Let us choose one of them, because spinor structures should not depend on the choice up to bundle equivalence. The metric, similarly as the KKS symplectic form, is invariant with respect to the coadjoint action of the group G. Therefore, the group G acts by isometries on the orbit $\mathcal{O}$. Hence the orbit can be treated further as a Riemannian homogeneous space of the form G/T.

As a consequence we can identify, accordingly to [20], the orthonormal frame bundle for the orbit with the following bundle:

$$\begin{array}{l} G/N \leftarrow T/N \\ \ \downarrow \\ M \simeq G/T. \end{array}$$

The next problem we should solve is the problem of derivation of the inefficiency kernel N for the coadjoint action.

Theorem. *Let G be a compact, connected, simple Lie group. Then the kernel of ineffectiveness N of the action of the group G in the space tangent at a point of flag manifold is isomorphic to* $N = T \cap Z(G)$*, where* T *is the maximal torus of G and* $Z(G)$ *is the centre of the group G.*

Proof. Let $\rho(g) = Ad^*_{g^{-1}}$, $g \in G$, be the coadjoint action $\rho(g)\colon \mathfrak{G}^* \to \mathfrak{G}^*$. Let $\mathcal{O}_\mu$ be the orbit of this action passing through $\mu \in \mathfrak{G}^*$. The orbit $\mathcal{O}_\mu$ can be identified with the coset space G/G_μ where $G_\mu = \{g \in G\colon \rho(g)\mu = \mu\}$ and we assume G_μ is a maximal torus in G. We look for a subgroup contained in G_μ, for which the action in the tangent space to the orbit induced by the coadjoint action is trivial. Let $v \in T_\mu\mathcal{O}_\mu$. Then the vector v is described by a curve $x(t) \subset \mathcal{O}_\mu$ such that $x(0) = \mu$, $\mathrm{d}x/\mathrm{d}t|_{t=0} = v$.

Then,

$$\forall \eta \in \mathfrak{G} \qquad g \in T \ \langle \rho(g)(x(t)), \eta \rangle = \langle Ad^*_{g^{-1}}(x(t)), \eta \rangle = \langle x(t), Ad_{g^{-1}}\eta \rangle.$$

We should calculate the derived map $\rho'(g)$ acting for $g \in T$ in the tangent space $T_\mu \mathcal{O}_\mu$. Let us calculate the derivative with respect to t at $t = 0$:

$$\langle \rho'(g)v, \eta \rangle = \langle v, Ad_{g^{-1}}\eta \rangle$$

but

$$\langle v, Ad_{g^{-1}}\eta \rangle = \langle Ad^*_{g^{-1}}v, \eta \rangle$$

therefore $\langle \rho'(g)v, \eta \rangle = \langle Ad^*_{g^{-1}}v, \eta \rangle$. Arbitrariness of $\eta \in \mathfrak{G}$ leads to

$$\rho'(g)v = Ad^*_{g^{-1}}v$$

where $g \in G_\mu$.

The subgroup N consists of those $g \in G$, which also belong to T and satisfy the condition

$$Ad^*_{g^{-1}}v = v \qquad \forall v \in T_\mu \mathcal{O}_\mu.$$

There exists a splitting $\mathfrak{G}^* = \mathfrak{H}^* \oplus \mathfrak{M}^*$ into two vector subspaces, where $\mathfrak{H}^*$ corresponds to the subgroup $G_\mu \simeq T$. Then $T_\mu \mathcal{O}_\mu \simeq \mathfrak{M}^*$ and our condition reads

$$\forall v \in \mathfrak{M}^* \qquad Ad^*_{g^{-1}}v = v.$$

Since $Ad^*_{g^{-1}}$ acts identically in $\mathfrak{H}^*$,

$$\forall h \in \mathfrak{H}^* \qquad v \in \mathfrak{M}^* \qquad Ad^*_{g^{-1}}(v + h) = v + h$$

this could be written as the condition

$$\forall \nu \in \mathfrak{G}^* \qquad Ad^*_{g^{-1}}\nu = \nu$$

because every element $\nu \in \mathfrak{G}^*$ could be represented in this form.

This condition considered for every $g \in G$ means $g \in Z(G)$ (the centre of G), for G connected but this is the case here. Taking into account that $g \in G_\mu$, this implies that the kernel of ineffectiveness of the action of the group G in the coadjoint orbit is

$$N = Z(G) \cap G_\mu = Z(G) \cap T \qquad \square$$

4. Spinor structures on the flag manifolds

There are at least two approaches to introducing spinors on flag manifolds. The reason for this is flag manifolds are Kähler (at least in our case of the finite-dimensional group G). The standard way, which is represented in this paper, is to construct the spinor structure double covering the orthonormal frame bundle and then the spinor bundle as a vector bundle associated with the spinor structure bundle by the Dirac representation. Spinor fields are smooth sections of the latter bundle. Topological obstacles to existence of the spinor structure are, as was mentioned above, the first and the second Stiefel–Whitney classes. In the other approach, geometric properties and, in particular, the existence of the almost complex structures on the generic coadjoint orbits make analysis of the structures easier, and, moreover, the construction of spinor fields more direct. Namely, existence of the almost complex structures implies orientability of the orbits and vanishing of the first Stiefel–Whitney class for them. As Freed [10] shows the problem of existence of the spinor bundle for these manifolds can be reformulated as the problem of existence of a square-root bundle of the standard geometric (pre)quantization bundle, existence of which is guaranteed if only the first Chern class of the tangent bundle of the orbit is integer. This is really the case as discussed Freed. Namely, $c_1(G/T) = 2\rho$, where ρ is the sum of the fundamental weights for G. Therefore, it is an integer. Moreover, it is even. This fact implies that there exists a quantization line bundle. Moreover, it implies in fact that there also exists a square-root bundle, i.e. the spinor bundle, because this integer is even. More formally, Freed proved that the second Stiefel–Whitney class w_2 is the reduction mod 2 of the first Chern class and, as a result, it vanishes. These considerations lead to the theorem, corollary 1.12 in [10]: 'G/T admits a spin structure'. It is also argued there that for every flag manifold there exists a unique spin structure because they are simply connected. We would like to construct the spin structures more explicitly, using the approach of Dąbrowski and Trautman [21, 20], and continue further with the standard approach to such constructions.

We will consider only the generic coadjoint orbits for non-exceptional simple compact groups. These groups are $SO(n)$, $SU(n)$ and $Sp(n)$. We will consider separately $SO(2k)$ and $SO(2k+1)$ groups, omitting only the trivial $SO(1)$ group and the cases of the $SO(2)$ (trivial orbit) and $SO(4)$ groups (this latter is not simple). The centres of these groups are:

$$\begin{aligned} &Z(SU(n)) = \mathbb{Z}_n \\ &Z(SO(2k)) = \mathbb{Z}_2 \\ &Z(SO(2k+1)) = \{e\} \\ &Z(Sp(n)) = \mathbb{Z}_2. \end{aligned} \tag{5}$$

The centres are contained in appropriate maximal tori and then we have the ineffectiveness kernels:

$$\begin{aligned} &N(SU(n)) = \mathbb{Z}_n \\ &N(SO(2k)) = \mathbb{Z}_2 \\ &N(SO(2k+1)) = \{e\} \\ &N(Sp(n)) = \mathbb{Z}_2. \end{aligned} \tag{6}$$

As the first result we state the structure of orthonormal frame bundles for all the orbits:

(a) $G = SU(n)$

$$\begin{array}{l} SU(n)/\mathbb{Z}_n \leftarrow T/\mathbb{Z}_n \\ \quad\downarrow \\ SU(n)/T. \end{array}$$

(b) $G = SO(2k)$

$$\begin{array}{l} SO(2k)/\mathbb{Z}_2 \leftarrow T/\mathbb{Z}_2 \\ \quad\downarrow \\ SO(2k)/T. \end{array}$$

(c) $G = SO(2k+1)$

$$\begin{array}{l} SO(2k+1) \leftarrow T \\ \quad\downarrow \\ SO(2k+1)/T. \end{array}$$

(d) $G = Sp(n)$

$$\begin{array}{l} Sp(n)/\mathbb{Z}_2 \leftarrow T/\mathbb{Z}_2 \\ \downarrow \\ Sp(n)/T. \end{array}$$

Let us consider next the universal covering bundles for total spaces of orthonormal frame bundles in the following particular cases.

(a) $G = SU(N)$, $N = \mathbb{Z}_n$, $G/N = SU(n)/\mathbb{Z}_n$.
The universal covering bundle for G/N is

$$\begin{array}{l} SU(n) \leftarrow \mathbb{Z}_n \\ \downarrow \\ SU(n)/\mathbb{Z}_n. \end{array} \tag{7}$$

(b) $G = SO(2k)$, $N = \mathbb{Z}_2$, $G/N = SO(2k)/\mathbb{Z}_2$.
The universal covering bundles for G/N are:

1.
$$\begin{array}{l} \mathrm{Spin}(2k) \leftarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{for} \quad k \text{ even} \\ \downarrow \\ SO(2k)/\mathbb{Z}_2 \end{array} \tag{8}$$

2.
$$\begin{array}{l} \mathrm{Spin}(2k) \leftarrow \mathbb{Z}_4 \qquad \text{for} \quad k \text{ odd} \\ \downarrow \\ SO(2k)/\mathbb{Z}_2. \end{array} \tag{9}$$

(c) $G = SO(2k+1)$, $N = \{e\}$, $G/N = SO(2k+1)$.
The universal covering bundle for G/N is

$$\begin{array}{l} \mathrm{Spin}(2k+1) \leftarrow \mathbb{Z}_2 \\ \downarrow \\ SO(2k+1). \end{array} \tag{10}$$

(d) $G = Sp(n)$, $N = \mathbb{Z}_2$, $G/N = Sp(n)/\mathbb{Z}_2$.
The universal covering bundle for G/N is

$$\begin{array}{l} Sp(n) \leftarrow \mathbb{Z}_2 \\ \downarrow \\ Sp(n)/\mathbb{Z}_2. \end{array} \tag{11}$$

Next we should consider homomorphisms for the appropriate groups $\Pi_1(G/N)$ to $\mathbb{Z}_2$, accordingly to the general scheme [20].

(a) $G = SU(n)$, $\Pi_1(G/N) = \mathbb{Z}_n$.
We should find homomorphisms $h: \mathbb{Z}_n \to \mathbb{Z}_2$, where

$$\mathbb{Z}_n = \left\{ \exp\left(\frac{\mathrm{i}2\pi}{n} k \right) \cdot \mathbb{1}: k = 0, 1, \ldots, n-1 \right\}.$$

These are:

1. h trivial—for all n
2. $h(\exp(\mathrm{i}2\pi/n)k\mathbb{1}) = (-\mathbb{1})^k$—for n even.

(b) $G = SO(2k)$, $k = 3, 4, \ldots$.

1. $G = SO(4l)$, $\Pi_1(SO(4l)/\mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2$.
   We look for homomorphisms $h: \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$. There exist four homomorphisms with the desired properties:
   (i) h—trivial.
   (ii) $h = pr_1$.
   (iii) $h = pr_2$.
   (iv) $h(a, b) = a \cdot b$.
   where pr_1, pr_2 are projections on the first and the second factor in the product, respectively.
2. $G = SO(4l + 2)$, $\Pi_1(SO(4l + 2)/\mathbb{Z}_2) = \mathbb{Z}_4$.
   We consider homomorphisms:
   $$h: \mathbb{Z}_4 \to \mathbb{Z}_2. \tag{12}$$
   There are two such homomorphisms:
   (i) h—trivial
   (ii) $h(a) = a^2$, where a is the generator of $\mathbb{Z}_4 = \{e, a, a^2, a^3\}$ and $a^4 = e$.

(c) $G = SO(2k + 1)$, $k = 2, 3, \ldots, N = \{e\}$. Since $\Pi_1(G/N) = \mathbb{Z}_2$ we should consider the homomorphisms $h: \mathbb{Z}_2 \to \mathbb{Z}_2$. There are two such homomorphisms:

1. h—trivial.
2. h—identity.

(d) $G = Sp(n)$, $\Pi_1(G/N) = \mathbb{Z}_2$. Similarly to the case (c) there are two homomorphisms $h: \Pi_1(G/N) \to \mathbb{Z}_2$:

1. h—trivial.
2. h—identity.

As the next step we should decide which homomorphisms lead to spinor structures for the orbits and construct the spinor structures as explicitly as possible.

(a) $G = SU(n)$, the orthonormal frame bundle for the orbit:

$$\begin{array}{l} SU(n)/\mathbb{Z}_n \leftarrow T/\mathbb{Z}_n \\ \downarrow \\ SU(n)/T. \end{array} \tag{13}$$

Let $n = 2k + 1$, then there is only one homomorphism:

$$h: \Pi_1(SU(2k + 1)/\mathbb{Z}_{2k+1}) \to \mathbb{Z}_2$$

which is the trivial homomorphism. Since there exists a spinor structure, which is unique, in this case the only possibility is

$$\begin{array}{l} (SU(2k + 1)/\mathbb{Z}_{2k+1}) \times \mathbb{Z}_2 \leftarrow (T/\mathbb{Z}_{2k+1}) \times \mathbb{Z}_2 \\ \downarrow \\ SU(2k + 1)/T. \end{array} \tag{14}$$

Let $n = 2k$. There are two homomorphisms:

$$h: \Pi_1(SU(2k)/\mathbb{Z}_{2k}) \to \mathbb{Z}_2.$$

One is trivial, the second one is given by

$$h\left(\exp\left(\frac{\mathrm{i}\pi}{k} l\right)\right) = (-\mathbb{1})^l \qquad l = 0, 1, \ldots, 2k - 1.$$

To obtain the spinor structure, let us take into consideration the following facts:

$$SU(n)/\mathbb{Z}_n = U(n)/U(1)$$

and the double cover for $SU(n)/\mathbb{Z}_n$ one obtains as the double cover for $U(n)/U(1)$. However, for $U(n)$ the double cover is the metaunitary group, which closes the following commutative diagram:

$$\begin{array}{ccc} MU(n) & \stackrel{inj}{\rightarrow} & \mathrm{Spin}(2n) \\ \rho_{2n}\downarrow & & \downarrow\rho_{2n} \\ U(n) & \stackrel{inj}{\rightarrow} & SO(2n). \end{array} \tag{15}$$

Therefore, the double cover of $U(n)/U(1)$ and therefore also of $SU(n)/\mathbb{Z}_n$ is $MU(n)/U(1)$.

In both cases of $SU(n)$ flag manifolds this should be the total space of the spinor structure bundle. We have to show in the case where $n = 2k+1$ a different description for the trivial homomorphism h. There is an alternative description of the acting group as $\hat{T}/(U(1)\times\mathbb{Z}_2)$ where $\hat{T}$ is the maximal torus in $MU(2k+1)$.

In the case where $n = 2k$ the trivial homomorphism does not lead to a spinor structure, but the non-trivial one does. The reason for this is that the trivial homomorphism leads to a structure group that is discrete, $(T/\mathbb{Z}_{2k})\times\mathbb{Z}_2$. Such a group could unambigously act on a total space of the bundle only in such a case in which the total space itself is a direct product by $\mathbb{Z}_2$ of some simply connected space. Otherwise, the base manifold would have to be not simply connected, which contradicts the well known and above discussed statement about all coadjoint orbits discussed in this paper. As a result, only the non-trivial homomorphism leads to a (as always unique) spinor structure. The diagram below summarizes the results for the maximal coadjoint orbits of $SU(n)$:

$$\begin{array}{l} MU(n)/U(1) \leftarrow \begin{cases} (T/\mathbb{Z}_{2k+1})\times\mathbb{Z}_2 & \text{for } n = 2k+1 \\ T/\mathbb{Z}_k & \text{for } n = 2k \end{cases} \\ \quad\downarrow \\ SU(n)/T. \end{array} \tag{16}$$

(b) $G = SO(4l)$, $\Pi_1(SO(4l)/\mathbb{Z}_2) = \mathbb{Z}_2\times\mathbb{Z}_2$.

A universally covering bundle of the total space of the orthonormal frame bundle:

$$\begin{array}{l} \mathrm{Spin}(4l) \leftarrow \mathbb{Z}_2^+\times\mathbb{Z}_2^- \\ \quad\downarrow \\ SO(4l)/\mathbb{Z}_2. \end{array} \tag{17}$$

There are four homomorphisms $h\colon \Pi_1(SO(4l)/\mathbb{Z}_2)\to\mathbb{Z}_2$:

1. h—trivial.
2. $h = pr_1$.
3. $h = pr_2$.
4. $h(a,b) = a\cdot b$.

A similar argument as in the case $SU(2k)$ shows that the trivial homomorphism cannot lead to a spinor structure. Cases 2 and 3 are also eliminated since they lead just to reductions of the universal covering bundles to one of the subgroups $\mathbb{Z}_2^+$ or $\mathbb{Z}_2^-$ of the structural group $\mathbb{Z}_2^+\times\mathbb{Z}_2^-$ of the universal cover of the total space of the orthonormal frame bundle. By elimination there is only one homomorhism left, that of case 4, leads to a spinor structure. The spinor structure is given by:

$$\begin{array}{l} \mathrm{Spin}(4l)/\mathbb{Z}_2 \leftarrow \hat{T}/\mathbb{Z}_2 \\ \quad\downarrow \\ SO(4l)/T \end{array} \tag{18}$$

where $\hat{T} = \rho_{4l}^{-1}(T)$, ρ_{4l} is the covering map.

(c) $G = SO(4l+2)$, $\Pi_1(SO(4l+2)/\mathbb{Z}_2) = \mathbb{Z}_4$.

There are two homomorphisms h: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2$.

1. h—trivial.
2. $h(a) = a^2$ for a the generator of $\mathbb{Z}_4$.

The universal covering bundle of the orthonormal frame bundle for the orbits:

$$\begin{array}{l} \mathrm{Spin}(4l+2) \leftarrow \mathbb{Z}_4 \\ \quad\downarrow \\ SO(4l+2)/\mathbb{Z}_2. \end{array} \tag{19}$$

By the same arguments as above we eliminate the trivial homomorhism as a candidate leading to a spinor structure in this case. The spinor structure on the orbit is connected with the non-trivial homomorphism.

It is given by

$$\begin{array}{l} \mathrm{Spin}(4l+2)/\mathbb{Z}_2 \leftarrow \hat{T}/\mathbb{Z}_2 \\ \quad\downarrow \\ SO(4l+2)/T \end{array} \tag{20}$$

$\hat{T} = \rho_{4l+2}^{-1}(T)$.

(d) $G = SO(2k+1)$, $\Pi_1(SO(2k+1)) = \mathbb{Z}_2$. The universal covering bundle for the orthonormal frame bundle over the orbit:

$$\begin{array}{l} \mathrm{Spin}(2k+1) \leftarrow \mathbb{Z}_2 \\ \quad\downarrow \\ SO(2k+1). \end{array} \tag{21}$$

Similarly to above, also in this case the trivial homomorphism cannot lead to a spinor structure. The spinor structure is connected with the identity homomorphism $h = \mathrm{i}d$ and is of the form

$$\begin{array}{l} \mathrm{Spin}(2k+1) \leftarrow \rho_{2k+1}^{-1}(T) \\ \quad\downarrow \\ SO(2k+1)/T \end{array} \tag{22}$$

ρ_{2k+1}: $\mathrm{Spin}(2k+1) \rightarrow SO(2k+1)$, the covering homomorphism.

(e) $G = Sp(n)$ $\quad \Pi_1(Sp(n)/\mathbb{Z}_2) = \mathbb{Z}_2$

The universal covering bundle of the orthonormal frame bundle over the orbit:

$$\begin{array}{l} Sp(n) \leftarrow \mathbb{Z}_2 \\ \quad\downarrow \\ Sp(n)/\mathbb{Z}_2. \end{array} \tag{23}$$

Once more, the trivial homomorphism is eliminated by the same argument as above as a candidate homomorphism which would lead to a spinor structure. The identity homomorphism $h = \mathrm{i}d$ gives the spinor structure for the orbit,

$$\begin{array}{l} Sp(n) \leftarrow T \\ \downarrow \\ Sp(n)/T. \end{array} \tag{24}$$

Examples

Let us consider some simple examples to see how the formalism works.

(a) $G = SU(2) \qquad T = \left\{ \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix} ; \phi \in [0, 2\pi[\right\} \simeq U(1)$

$G/T = SU(2)/U(1) \simeq S^2$. Our formalism gives the orthonormal frame bundle over S^2:

$$\begin{array}{l} SU(2)/\mathbb{Z}_2 \leftarrow U(1)/\mathbb{Z}_2 \\ \downarrow \\ S^2 \end{array} \tag{25}$$

by standard isomorphisms this bundle is identical to

$$\begin{array}{l} SO(3) \leftarrow SO(2) \\ \downarrow \\ S^2 \end{array} \tag{26}$$

which is the standard orthonormal frame bundle for the two-dimensional sphere. We proceed further within the general scheme introduced above.

We should find first the metaunitary group $MU(2)$ double covering the group $U(2)$, considered as a subgroup in $SO(4)$. Therefore, we look for the group $MU(2)$ such that the diagram

$$\begin{array}{ccc} MU(2) & \overset{inj}{\rightarrow} & \mathrm{Spin}(4) \\ \rho_4 \downarrow & & \downarrow \rho_4 \\ U(2) & \overset{inj}{\rightarrow} & SO(4) \end{array} \tag{27}$$

is commutative. The group $\mathrm{Spin}(4) = SU(2) \times SU(2)$. As is known $MU(2) = SU(2) \times U(1)$ and the injection is given by a simple product of the identity map from $SU(2)$ to itself and of the injection map from $U(1)$, treated as a subgroup of the second $SU(2)$ group in the $\mathrm{Spin}(4)$ group, to this group.

The spinor structure on the S^2 treated as a coadjoint orbit of the $SU(2)$ group is then given generally by

$$\begin{array}{l} MU(2)/U(1) \leftarrow U(1)/\{e\} \\ \downarrow \\ S^2. \end{array} \tag{28}$$

Since $MU(2) = SU(2) \times U(1)$, $MU(2)/U(1) = SU(2)$ the spinor structure is given by

$$\begin{array}{l} SU(2) \leftarrow U(1) \\ \downarrow \\ S^2. \end{array} \tag{29}$$

Since $SU(2) \simeq \mathrm{Spin}(3)$, $U(1) \simeq \mathrm{Spin}(2)$, the spinor structure can be represented as

$$\begin{array}{l}\mathrm{Spin}(3) \leftarrow \mathrm{Spin}(2)\\ \quad\downarrow \\ \quad S^2\end{array} \tag{30}$$

which is the standard spinor structure on S^2.

(b) $G = SO(3)$ $\qquad T = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} : \phi \in [0, 2\pi[\right\} \simeq SO(2).$

The flag manifold is equivalent to $SO(3)/SO(2) \simeq S^2$ and so the orbit is the same manifold as in the first example. The orthonormal frame bundle looks, accordingly to the general scheme, as follows:

$$\begin{array}{l}SO(3) \leftarrow SO(2)\\ \quad\downarrow \\ \quad S^2\end{array} \tag{31}$$

and this is the standard orthonormal frame bundle over the two-dimensional sphere. We expect then that the general procedure should give the standard spinor structure over S^2. Let us check this. The spinor structure should be of the form:

$$\begin{array}{l}\mathrm{Spin}(3) \leftarrow \rho_3^{-1}(T)\\ \quad\downarrow \\ \quad S^2.\end{array} \tag{32}$$

Since T is isomorphic to $SO(2)$ embedded in $SO(3)$ in the way shown above, $\rho_3^{-1}(T)$ is the subgroup isomorphic to $\mathrm{Spin}(2)$ embedded in $\mathrm{Spin}(3)$ in the standard way. Therefore, the spinor structure could be written as

$$\begin{array}{l}\mathrm{Spin}(3) \leftarrow \mathrm{Spin}(2)\\ \quad\downarrow \\ \quad S^2\end{array} \tag{33}$$

and this is once more the standard spinor structure for the two-dimensional sphere.

5. Conclusions

We constructed as explicitly as possible spinor structures on flag manifolds of non-exceptional simple compact groups. It is worth mentioning that the construction we have presented also gives us for free the (reduced) symplectic spinor structure. The reason is the symplectic structure agrees with the almost complex structure. Nevertheless, the generalization of our results to the latter groups is straightforward. In our future papers we would like to investigate spinor bundles and spinor fields over the orbits. In particular, we are interested in spinor fields satisfying the Dirac equation. This research would be a starting point for more involved investigations of the spinor fields over coadjoint orbits of groups *SDiff M*. Whether existence of these structures is connected with conditions for polarization of the orbits found by Goldin *et al* is the most exciting problem which we would like to investigate.

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